

Low Space External Memory Construction of the Succinct Permuted Longest Common Prefix Array

German Tischler

Max Planck Institute of Molecular Cell Biology and Genetics, Pfotenhauerstraße 108,
01037 Dresden, Germany
tischler@mpi-cbg.de

Abstract. The longest common prefix (LCP) array is a versatile auxiliary data structure in indexed string matching. It can be used to speed up searching using the suffix array (SA) and provides an implicit representation of the topology of an underlying suffix tree. The LCP array of a string of length n can be represented as an array of length n words, or, in the presence of the SA, as a bit vector of $2n$ bits plus asymptotically negligible support data structures. External memory construction algorithms for the LCP array have been proposed, but those proposed so far have a space requirement of $O(n)$ words (i.e. $O(n \log n)$ bits) in external memory. This space requirement is in some practical cases prohibitively expensive. We present an external memory algorithm for constructing the $2n$ bit version of the LCP array which uses $O(n \log \sigma)$ bits of additional space in external memory when given a (compressed) BWT with alphabet size σ and a sampled inverse suffix array at sampling rate $O(\log n)$. This is often a significant space gain in practice where σ is usually much smaller than n or even constant. We also consider the case of computing succinct LCP arrays for circular strings.

1 Introduction

The suffix array (SA) and longest common prefix array (LCP) were introduced as a lower memory variant of the suffix tree (cf. [27]) for exact string matching using a pre computed index (cf. [19]). For a text of length n both can be computed in linear time in internal memory (IM) (cf. [17,18,5]) and require n words of memory each. For large texts the space requirements of SA and LCP in IM can be prohibitive. Compressed and succinct variants including compressed suffix arrays (see e.g. [13,22,12]), the FM index and variants (see [9,10,11]) and succinct LCP arrays (see [23]) use less space, but for practicality it is also crucial to be able to construct these data structures using affordable space requirements. Construction algorithms for compressed suffix arrays and the Burrows Wheeler transform (BWT, see [4]) using $o(n \log n)$ bits of space in IM (assuming $\sigma \in o(n)$) were introduced (see e.g. [14,21]). It is still unclear whether these algorithms scale well in practice. At the very least they require an amount of IM which

is several times larger than what is needed for the input text. External memory solutions for constructing the suffix array and LCP array have also been presented (see e.g. [3,6,15]). These algorithms require $O(n)$ words ($O(n \log n)$ bits) of external memory (EM). However, as for their IM pendants, this space requirement is large if the algorithms are used as a vehicle to obtain a compressed representation. Recently algorithms for constructing the BWT in EM without explicitly constructing a full suffix array were designed and implemented (see [8,25]). In this paper we present an algorithm for constructing a succinct LCP array in EM based on a BWT and sampled inverse suffix array while using $O(n \log \sigma)$ instead of $O(n \log n)$ bits of space in EM. Both, BWT and sampled inverse suffix array can be produced in space $O(n \log \sigma)$ in external memory by the algorithm presented in [25,26]. In the final section of this paper we consider the extension of our algorithm to circular strings.

2 Definitions

Let Σ denote a totally ordered and ranked alphabet, w.l.o.g. we assume $\Sigma = \{0, 1, \dots, \sigma - 1\}$ for some $\sigma > 0$. Further let $s = s_0 s_1 \dots s_{n-1}$ denote a string of length $|s| = n > 0$ over Σ s.t. the last symbol of s is the minimal symbol in s and does not appear elsewhere in s . We use $s[i]$ to denote s_i and $s[i..j]$ for $s_i s_{i+1} \dots s_j$ for $0 \leq i \leq j < n$. $s[i..j]$ denotes the empty string for $i > j$. The i 'th suffix of s denoted by \tilde{s}_i is the string $s[i..n-1]$. Suffix \tilde{s}_i is smaller than \tilde{s}_j for $i \neq j$ (denoted by $\tilde{s}_i < \tilde{s}_j$) if for the smallest k s.t. $s[i+k] \neq s[j+k]$ we have $s[i+k] < s[j+k]$. The suffix array SA of s is the permutation of $0, 1, \dots, n-1$ s.t. $\tilde{s}_{\text{SA}[i-1]} < \tilde{s}_{\text{SA}[i]}$ for $i = 1, 2, \dots, n-1$. For two suffixes \tilde{s}_i and \tilde{s}_j with $i \neq j$ the longest common prefix $\text{lcp}(i, j)$ of the two is $s[i..i+\ell-1]$ for the smallest ℓ s.t. $s[i+\ell] \neq s[j+\ell]$. The array LCP of s is defined by $\text{LCP}[i] = |\text{lcp}(\text{SA}[i-1], \text{SA}[i])|$ for $i > 0$ and $\text{LCP}[0] = 0$. The inverse suffix array ISA of s is defined by $\text{ISA}[\text{SA}[i]] = i$ for $0 \leq i < n$. The permuted LCP array PLCP of s is given by $\text{PLCP}[i] = \text{LCP}[\text{ISA}[i]]$ for $0 \leq i < n$ and $\text{PLCP}[i] = 0$ otherwise. The Burrows Wheeler transform BWT of s is defined by $\text{BWT}[i] = s[(\text{SA}[i] + n - 1) \bmod n]$ for $0 \leq i < n$. Let C be the array of length σ s.t. $C[a] = |\{i \mid s[i] = a\}|$ for $a \in \Sigma$ and let D be an array of length $\sigma + 1$ s.t. $D[a] = \sum_{i < a} C[i]$ for $0 \leq a \leq \sigma$. For a sequence $t = t_0, t_1, \dots, t_{k-1}$ for some $k \geq 0$ let $\text{RANK}_t(a, j) = |\{i \mid 0 \leq i < \min(j, k), t_i = a\}|$, i.e. the number of a elements in t up to but excluding index j and let $\text{SELECT}_t(a, j) = \min\{i \mid \text{RANK}_t(a, i+1) = j+1\}$ if $0 \leq j < \text{RANK}_t(a, k)$ and undefined otherwise. LF is defined by $\text{LF}(r) = \text{ISA}[(\text{SA}[r] + n - 1) \bmod n]$. B is defined by $\text{B}(a, i) = D[a] + \text{RANK}_{\text{BWT}}(a, i)$ for $a \in \sigma, 0 \leq i \leq n$ and

BACKSTEP by $\text{BACKSTEP}(a, (i, j)) = (B(a, i), B(a, j))$ for $a \in \Sigma, 0 \leq i, j \leq n$.

3 Previous Work

The first linear time algorithm for computing the LCP array from the suffix array and text appeared in [18]. One of the main combinatorial properties used by this algorithm is the fact that $\text{PLCP}[i] \geq \text{PLCP}[i-1] - 1$ for $0 < i < n$. This property is also used in [23] to obtain a representation of the PLCP array using $2n + o(n)$ bits while allowing constant time access. Let $\zeta(0) = 1$ and $\zeta(i) = 0\zeta(i-1)$ for $i > 0$. The $2n$ bits in the data structure are the bit sequence $K = \eta(n-1)$ given by $\eta(0) = \zeta(\text{PLCP}[0] + 1)$ and $\eta(i) = \eta(i-1)\zeta(\text{PLCP}[i] - \text{PLCP}[i-1] + 1)$ for $0 < i < n$. The $o(n)$ additional bits are used for a select index (cf.[20]) on K . K stores the sequence of pairwise differences of adjacent PLCP values shifted by 1 in unary representation (the number i is represented as i zero bits followed by a 1 bit). The value $\text{PLCP}[i]$ can be retrieved as $\text{SELECT}_K(1, i) - 2(i+1) - 1$. In [2] Beller et al present an algorithm for computing the LCP array in IM using a wavelet tree (see [12]). This algorithm runs for $\ell_m + 1$ rounds where ℓ_m is the maximum LCP value produced. Round i for $0 \leq i \leq \ell_m$ sets $\text{LCP}[r]$ for exactly those ranks r s.t. $\text{LCP}[r] = i$, i.e. the values are produced in increasing order.

4 Computing the succinct PLCP array

In this section we modify the algorithm by Beller et al (cf. [2]) to produce the succinct $2n$ bit PLCP bit vector in EM. The main idea is to use the fact that the algorithm produces the LCP values in increasing order. It starts with a tuple $(\epsilon, (0, n))$ which denotes the empty word and the corresponding rank interval on the suffix array (the lower end 0 is included, the upper n is excluded). Round i takes the tuples from the previous round (or the start tuple for round 0) and considers all possible extensions by one symbol via backward search (cf. [9]), i.e. it produces $(aw, (l', r'))$ from $(w, (l, r))$ for each aw appearing in s . All suffixes considered in round i starting by aw in the rank interval (l', r') have a common prefix of length $i+1$, while the suffixes at ranks $l' - 1$ and l' (for $l \neq 0$) as well as at ranks $r' - 1$ and r' (for $r' < n$) have a common prefix of at most length i . Based on this insight we can set $\text{LCP}[l']$ and $\text{LCP}[r']$ to i , if they have not already been set in a previous round. In the tuples the first (string) component is only provided for the sake of exposition, the algorithm does not require or use it. In addition the algorithm prunes away intervals when a respective LCP

value (Beller et al use the upper bound r' for setting new values in [2], we in this paper use the lower bound l' as it simplifies the transition to EM) is already set.

The succinct PLCP array K contains n zero and n one bits. The one bits mark positions in the text (remember PLCP is in text order). The zero bits encode the differences between adjacent PLCP values shifted by 1. For computing this bit vector assume that we start off with a vector of n one bits. The information we need in addition is in front of which 1 bit we have to insert how many 0 bits. If $\text{PLCP}[i]$ is not smaller than $\text{PLCP}[i - 1]$, then we have to add $\text{PLCP}[i] - \text{PLCP}[i - 1] + 1$ zero bits just in front of the $i + 1$ st 1 bit. In the algorithm we can achieve this by starting to add 0 bits for ranks which did not have their value set in a previous round but which do have the value for the rank of the previous position set in the current round. We call this adding a rank to the active set. We stop adding 0 bits for a rank in the round in which the value for the rank itself gets set, which we call removing a rank from the active set. Figure 1 shows an algorithm implementing this approach in IM. A wavelet tree (cf. [12]) for BWT can be used to compute the BACKSTEP, RANK and SELECT functions in time $O(\log \sigma)$ and to determine the set of symbols occurring in any index interval on BWT in time $O(\log \sigma + o)$ where o is the number of distinct symbols in the interval.

In the following we show how to adapt this algorithm so it becomes usable in EM and requires no more than $O(n \log \sigma)$ space in EM while using $O(\sigma \log n)$ bits of IM. This means we need to make sure that all data structures used in EM are accessed in a purely sequential way and none use $\omega(n \log \sigma)$ space. In particular we need to consider the representation and access patterns of the queues Q and NQ , the Burrows Wheeler transform BWT of s , the sets S , T and `activeSet` and the counter array for zero bits `PD`.

For some of the representations we will use Elias γ code (cf. [7]) and the following result proven in [26].

Lemma 1 ([26]). *Let G denote an array of length ℓ such that $G[i] \in \mathbb{N}$ for $0 \leq i < \ell$ and $\sum_{i=0}^{\ell-1} G[i] = s$ for some $s \in \mathbb{N}$. Then the γ code for G takes $O(\ell + s)$ bits.*

This means we can represent any strictly increasing sequence x_0, x_1, \dots, x_{k-1} of numbers from $0, 1, \dots, N$ for $N \in O(n)$ and $k > 0, k \in O(n)$ in $O(n)$ bits by storing the differences $x_i - x_{i-1}$ for $i = 0, 1, \dots, k - 1$ in γ code where we assume $x_{-1} = -1$.

- The queue NQ is not produced in increasing order in the algorithm as stated in Figure 1 (meaning if (l_1, r_1) is enqueued right after (l_0, r_0) then we cannot assume $l_1 \geq r_0$). If however the queue Q is in increasing order

```

PLCPINTERNAL(BWT, n, ISA)
1  (Q, activeSet, S, PD)  $\leftarrow$  ( $\emptyset$ ,  $\emptyset$ ,  $\emptyset$ ,  $\emptyset$ )
2  Q.ENQUEUE((0, n))
3  while Q.EMPTY() = FALSE do
4       $\triangleright$  Queue for next round and ranks set in this round
5      (NQ, T)  $\leftarrow$  ( $\emptyset$ ,  $\emptyset$ )
6      while Q.HASNEXT() do
7          (l, r)  $\leftarrow$  Q.NEXT()
8          foreach sym  $\in$  {BWT[i] |  $1 \leq i < r$ } do
9              (l', r')  $\leftarrow$  BACKSTEP(sym, (l, r))
10             if S.CONTAINS(l') = FALSE then
11                  $\triangleright$  get l'src s.t. LF(l'src)=l'
12                  $\triangleright$  this is the smallest i s.t.  $l \leq i < r$  and BWT[i]=sym
13                 l'src  $\leftarrow$  SELECTBWT(sym, RANKBWT(sym, l))
14                  $\triangleright$  mark l' as to be set in this round
15                 T.INSERT(l')
16                  $\triangleright$  put l'src in active set if not set yet
17                 if S.CONTAINS(l'src) = FALSE then
18                     activeSet.INSERT(l'src)
19                     NQ.ENQUEUE((l', r'))
20              $\triangleright$  Increment number of 0 bits for ranks in active set
21             foreach r  $\in$  activeSet do
22                 if PD.CONTAINS(r) then
23                     PD[r]  $\leftarrow$  PD[r] + 1
24                 else PD[r]  $\leftarrow$  1
25              $\triangleright$  Remove ranks set in this round from active list
26              $\triangleright$  and update set of ranks finished
27             foreach r  $\in$  T do
28                 if activeSet.CONTAINS(r) then
29                     activeSet.REMOVE(r)
30             S.INSERT(r)
31     Q  $\leftarrow$  NQ
32  $\triangleright$  Produce succinct bit vector in text order
33 (i, K)  $\leftarrow$  (0,  $\emptyset$ )
34 for p  $\leftarrow$  0 to n - 1 do
35     r  $\leftarrow$  ISA[p]
36     if PD.CONTAINS(r) then
37         for j  $\leftarrow$  1 to PD[r] do
38             K[i++]  $\leftarrow$  FALSE
39     K[i++]  $\leftarrow$  TRUE
40 return B

```

Fig. 1. Internal memory version of PLCP computation algorithm

and we consider only the intervals produced by extensions with a fixed symbol **sym**, then those extension intervals are in increasing order. The **B** function is for a fixed first argument **sym** monotonously increasing in its second argument and has a maximum value of $D[\mathbf{sym} + 1]$ which is only reached as an (excluded) right end of any **BACKSTEP** call and at the same time the (included) minimum left end of calls for **BACKSTEP** with first parameter **sym+1** (if any such exist in s). This means if we replace **NQ.enqueue**((l', r')) by **NQ.ENQUEUE**(**sym**, (l', r')), sort **NQ** stably by the first (**sym**) component and subsequently drop the first component then the resulting list of intervals will be in sorted order. The sorting can be performed using $O(\log \sigma)$ rounds of bucket sorting along the bit representation of the first component, each of which takes $O(n)$ time as we can never have more than n elements in the queue. During the whole sorting procedure the elements for each single first component will stay in ascending order concerning their second component, which allows us to store the second component using differential γ code. The sequence of lower interval bounds and the one of upper interval bounds both form strictly increasing sequences. Starting the difference coding for the sequences for **sym** at $D[\mathbf{sym}] - 1$ ensures that for both sequences the sum of the stored numbers does not exceed n , so we can store them using $O(n)$ bits according to Lemma 1.

- The **T** set stores a subset of the lower interval bounds produced for **NQ**. We can thus use similar steps to produce it in sorted order while requiring $O(n \log \sigma)$ bits of space in **EM** and $O(\sigma \log n)$ bit of **IM**.
- The values added to **activeSet** in line 18 can easily be added in increasing order by first storing them in a heap data structure for each source interval (l, r) and writing the values out in order at the end of the handling of (l, r). This takes space $O(\sigma \log n)$ in **IM** while the run time for this is bounded by $O(n \log \sigma)$ for each round (the heap depth is bounded by $\log \sigma$ as we never insert more than σ elements into any heap and the total number of elements added is bounded by n). The values in increasing order can again be stored using differential γ code in $O(n)$ bits. As soon as we have the set of newly added values for a round we can merge it into the set of previously added values, which can be stored in the same way. Storing **activeSet** in this way requires $O(n)$ bits of space in **EM**.
- For each source interval (l, r) the set of symbols in $\{\mathbf{BWT}[i] \mid 1 \leq i < r\}$, the target intervals (l', r') and the respective l'_{src} values can be computed during a linear scan of the **BWT** sequence streamed from **EM** while keeping track of the values of the **RANK** function for each symbol. This requires $O(\sigma \log n)$ bits of space in **IM**. We keep tu-

```

BINUNBUCKETSORT( $\mathcal{K}$ ,  $A$ ,  $m$ )
1  ( $\text{cnt}[0], \text{cnt}[1]$ )  $\leftarrow$  (0, 0)
2  for  $i \leftarrow 0$  to  $m - 1$  do
3       $\text{cnt}[\mathcal{K}[i]] \leftarrow \text{cnt}[\mathcal{K}[i]] + 1$ 
4  ( $\text{cnt}[0], \text{cnt}[1]$ )  $\leftarrow$  (0,  $\text{cnt}[1]$ )
5  for  $i \leftarrow 0$  to  $m - 1$  do
6       $B[i] \leftarrow A[\text{cnt}[\mathcal{K}[i]]]$ 
7       $\text{cnt}[\mathcal{K}[i]] \leftarrow \text{cnt}[\mathcal{K}[i]] + 1$ 
8  return  $B$ 

```

Fig. 2. Inverse binary bucket sorting for key vector \mathcal{K} and data vector A , both of length m

ples $(\text{sym}, l', \hat{r}, l' \text{src})$ in an AVL tree (cf. [1]) where only the first (sym) component is used as the key. While scanning BWT we insert $(\text{sym}, B(\text{sym}, l' \text{src}), B(\text{sym}, l' \text{src}) + 1, l' \text{src})$ upon first encountering sym at index $l' \text{src}$ in (l, r) and update the third component accordingly whenever we find another instance of sym in the source interval. With the same reasoning as above for the heap used while handling `activeSet` this takes time $O(n \log \sigma)$ for one round.

- The accesses to S in line 17 are in ascending index order and updating S in line 30 while scanning S and T can read both sequences in linear ascending order, which is suitable for EM. Accessing S at l' in line 10 is somewhat more challenging. As shown above the l' values in each round are only increasing when we look at a single symbol sym . We can obtain the bits we need to see in the required order using the following steps. First compute the sequence of l' values we need to access in ascending order. This can be done as described above for producing NQ, i.e. produce a set of pairs (sym, l') , sort it by the first component while using differential γ code for representing the second components and then drop the first component. This takes time $O(n \log \sigma)$ and space $O(n \log \sigma)$ bits in EM. It gives us the set of required l' values in increasing order and thus makes it easy to determine whether S does or does not contain the respective values, which we store as a bit vector in EM. This bit vector has as many bits as l' values relevant in the current round, which is $O(n)$. Now we have the relevant bits, but they are in the wrong order, as we sorted the l' values by the respective sym values. We can reorder the bits by *inverse sorting* them using the original order of the sym values. Figure 2 shows an algorithm which performs inverse sorting of a sequence A given a binary key vector \mathcal{K} . It does this by first determining how many 0 and 1 bits there are in the key vector (lines 1-3) and then rebuilding the original sequence by scanning \mathcal{K} and taking elements from the 0 and 1 regions of the sorted

sequence in accordance with the key bits encountered (lines 5-7). This inverse binary bucket sorting can be extended to inverse radix sorting for non binary keys. It requires time $O(n \log \sigma)$ (we need $\log \sigma$ rounds of inverse bucket sorting) and space $O(n \log \sigma)$ bits in EM.

- The PD array can be represented as a bit vector in EM. We initialise it as a vector of n one bits. Adding one to index r is done by inserting a zero bit just ahead of the $k + 1$ 'st one bit. We scan `activeSet` and PD linearly for updating PD where PD has at most $2n$ bits at any time. So updating PD in each round takes $O(n)$ time and storing PD takes $O(n)$ bits in EM.

Overall each round of the algorithm up to line 32 takes time $O(n \log \sigma)$ and we need $O(n \log \sigma)$ bits of space in EM. In the worst case the maximum LCP value is $n - 2$ (which is e.g. reached $s[i] = 1$ for $0 \leq i < n - 1$ and $s[n - 1] = 0$), so the worst case run time of the algorithm is $O(n^2 \log \sigma)$. In the average case (cf. [24]) the maximum value is in $O(\log_\sigma n)$, which gives this part of the algorithm a run time of $O(n \log n \log \sigma)$ on average.

This leaves us with the issue that the procedure above so far produces the difference between PLCP values in rank instead of position order. This is set right by lines 33-39 in Figure 1, however it uses a complete inverse suffix array and requires random access to the PD array. Given a sampled inverse suffix array at sampling rate $f \in O(\log n)$ taking $O(n)$ bits, the BWT and the PD bit vector we can produce the final PLCP bit vector using the following steps:

1. Create pairs $(\text{ISA}[if], if)$ for $i = 0, 1, \dots, \lceil \frac{n}{f} \rceil - 1$ in EM from the sampled inverse suffix array (both components are stored as $O(\log n)$ bit block code) and sort these pairs by their first (rank) component using radix sort. This takes space $O(n)$ bits in EM and time $O(\frac{n}{f} \log n) = O(n)$. After sorting annotate each tuple with one bit set to `true` as third component (marks the tuple as active), the number 0 stored in γ code as fourth component (stores the number of PLCP values added to the tuple so far) and an empty vector of γ coded numbers as the fifth component.
2. For f rounds do the following: perform an LF operation on the tuples (map (r, p, a, b, c) to $(\text{BWT}[r], \text{LF}(r), p', a, b, c)$ where $p' = (p + n - 1) \bmod n$ if a is `true` and p otherwise) by scanning the BWT and computing LF as described above while tracking the B function using $O(\sigma \log n)$ bits of IM. Sort the resulting tuples by the first component and drop the first component. This restores the sorted order according to the rank of the tuples and takes time $O(n \log \sigma)$ and space $O(n \log \sigma)$ bits in EM. Note that for each active (third component is `true`) tuple in the list we

- retain the invariant that for a first component r we have $\text{SA}[r]$ as the second component. Scan the tuples and the PD bit vector and copy the respective (matching rank) values into tuples marked as active by inserting the value $\text{PD}(r)$ at the front of the vector of γ coded values in component five and incrementing the counter for appended values (fourth component) by one. This takes time $O(n)$ and again space $O(n \log \sigma)$ bits in EM. In another scan mark tuples s.t. their second component p is divided by f as inactive. Note that at the end of each round we have the following property: Let $(r, p, a, c, (v_0, v_1, \dots, v_{c-1}))$ be a tuple in our list. Then for $i = 0, 1, \dots, c-1$ we have $v_i = \text{PLCP}[p+i] - \text{PLCP}[p+i-1] + 1$.
3. Sort the tuples by the second component (position) using a log n round radix sort taking $O(n)$ time and $O(n \log \sigma)$ bits of space. Let $t_0, t_1, \dots, t_{\lceil \frac{n}{f} \rceil - 1}$ be the sequence of tuples we have obtained. Then for each $t_i = (r, p, a, b, c)$ with $0 \leq i \leq \lceil \frac{n}{f} \rceil - 1$ we now have $p = if$, $a = \text{false}$, b represents $\min(n - p, f)$ and c is the sequence v_0, v_1, \dots, v_{b-1} s.t. $v_i = \text{PLCP}[p+i] - \text{PLCP}[p+i-1] + 1$.
 4. Initialise an empty bit vector K . Scan the tuples and for each tuple do the following: let c denote the number stored in the fourth component and let v_0, v_1, \dots, v_{c-1} be the (decoded) numbers stored in the fifth component. For i in $0, 1, \dots, c-1$ append v_i zero bits to K and then 1 one bit.

The bit vector K is by construction the succinct $2n$ bit representation of the PLCP array. The whole reordering takes time $O(n \log n \log \sigma)$, $O(n \log \sigma)$ bits of space in EM and $O(\sigma \log n)$ bits of space in IM. Each tuple at maximum uses $\log \sigma$ bits for the symbol intermediately introduced in step 2, $O(\log n)$ bits for rank and position and $O(\log n)$ bits for storing the number of PD values copied into the tuple so far. The sum over all stored γ values in the last component of the tuples is bounded by n and reaches n at the end of the procedure.

We summarise the run time and space requirements of the EM algorithm in the following Theorem.

Theorem 1. *The succinct $2n$ bit PLCP representation for a string s of length n can, given it's BWT and sampled suffix array of sampling rate $f \in O(\log n)$, be constructed in worst cast time $O(n^2 \log \sigma)$ and average time $O(n \log n \log \sigma)$ using $O(n \log \sigma)$ bits of space in EM and $O(\sigma \log n)$ bits of space in IM.*

5 Reducing Internal Memory Usage

While the algorithm of the previous section has space requirements in $O(n \log \sigma)$ bits in external memory, the need for $O(\sigma \log n)$ bits in IM may

be considered as too large in some situations, even though it is not an obstacle in practice. We can modify the algorithm to use less space in internal memory, as we show in the following. A suitable reformulation of the algorithm is given in Figure 3. The algorithm as shown only reformulates the computation of the bit vector up to the point where it is translated from rank to position order. The crucial point about the reformulation is to compute the LF and BACKSTEP functions without keeping track of the value of the RANK function in IM for each single symbol in Σ . Observe that given a set of ranks R we can compute the set of ranks R_{LF} defined by $R_{LF} = \{r' \mid r' = LF(r), r \in R\}$ using the following steps: create a bit vector R_B of length n s.t. $R_{B_r} = 1$ iff $r \in R$ and then construct the sequence of pairs $P_R = (BWT_0, R_{B_0})(BWT_1, R_{B_1}) \dots (BWT_{n-1}, R_{B_{n-1}})$. Sort P_R by the first (symbol) component in a stable way using radix sort in time $O(n \log \sigma)$. It is easy to see that the second (bit) component of the sorted vector represents R_{LF} by virtue of marking the respective ranks by 1 bits. This method can be extended to computing the BACKSTEP function for a given set of intervals and all possible extensions of the respective intervals on the left. To this end observe that for a given interval $[l, r)$ an extension is possible by exactly those symbols contained in the set given by $\{a \mid a = BWT_i \text{ for some } l \leq i < r\}$, the lower bound l' of $(l', r') = \text{BACKSTEP}(a, (l, r))$ for any such symbol is given by $l' = LF(l_{src})$ where l' is the smallest number s.t. $l \leq l_{src} < r$ and $BWT_{l_{src}} = a$ and $r' - l'$ equals the number of a symbols in the sequence $BWT_l, BWT_{l+1}, \dots, BWT_{r-1}$. The depicted algorithm computes all extensions of a given set of intervals by the BACKSTEP function using the following steps. Assume a list of intervals $L = (l_0, r_0), (l_1, r_1) \dots, (l_{m-1}, r_{m-1})$ is given s.t. $l_0 = 0$, $r_{i-1} = l_i$ for $i = 1, 2, \dots, m-1$ and $r_{m-1} = n$. In particular the intervals partition the index space $0, 1, \dots, n-1$. L can be stored using $O(n)$ bits in external memory using either γ code for storing the increasing sequences of lower and upper bounds using differential encoding or by storing two bit vectors of length n marking the start and end of the intervals. For each interval (l_i, r_i) in ascending order do the following to produce a sequence \mathcal{Z} :

1. extract the sequence $B = BWT_{l_i}, BWT_{l_i+1}, \dots, BWT_{r_i-1}$ to B_S and sort it in time $O((r-l) \log \sigma)$ using radix sort
2. in a single linear scan of B_S mark the first occurrence of symbol a in B_S with the number of times it occurs in B_S , i.e. $|\{i \mid 0 \leq i < r-l \text{ and } B_{S_i} = a\}| = |\{i \mid 0 \leq i < r-l \text{ and } B_i = a\}| = |\{i \mid l \leq i < r \text{ and } BWT_i = a\}|$. The rest of the character instances are marked with zero. The attached numbers are stored using γ code. The numbers stored obviously sum up to $r-l$. Let the obtained sequence be called B_M .

3. append B_M to \mathcal{Z} .

Then sort \mathcal{Z} stably by the first (symbol) component using radix sort in time $O(n \log \sigma)$. Let $\mathcal{Z}_S = (a_0, v_0), (a_1, v_1), \dots, (a_n, v_n)$ denote the resulting sorted sequence. Further let $J = \{j \mid v_j \neq 0\} = j_0, j_1, \dots, j_{k-1}$ and $I = (j_0, v_{j_0}), (j_1, v_{j_1}), \dots, (j_{k-1}, v_{j_{k-1}})$. Let

$$\text{BACKSTEP}^*(a, L) = \text{BACKSTEP}(a, (l_0, r_0)), \dots, \text{BACKSTEP}(a, (l_{m-1}, r_{m-1}))$$

for $a \in \Sigma$ and

$$\text{BACKSTEP}'(L) = \text{BACKSTEP}^*(0, L), \dots, \text{BACKSTEP}^*(\sigma - 1, L) .$$

Let the filter function FLT be defined by

$$\text{FLT}((\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_z, \beta_z)) = \begin{cases} (\alpha_1, \beta_1), \text{FLT}((\alpha_2, \beta_2), \dots, (\alpha_z, \beta_z)) & \text{if } \alpha_1 \neq \beta_1 \\ \text{FLT}((\alpha_2, \beta_2), \dots, (\alpha_z, \beta_z)) & \text{otherwise} \end{cases}$$

Following the same pattern as computing the LF function by attaching the BWT symbols to a bit vector it is straight forward to see that I is exactly the sequence of intervals $\text{FLT}(\text{BACKSTEP}'(L))$, i.e. all non empty extensions of intervals in L in ascending order. In consequence we obtain the following result.

Lemma 2. *Given BWT and a sorted, non overlapping list of intervals L drawn from $[0, n)$ s.t. both BWT and L can be decoded in constant time per element the sorted sequence of intervals $\text{FLT}(\text{BACKSTEP}'(L))$ can be computed in time $O(n \log \sigma)$ and space $O(n \log \sigma)$ bits in EM and $O(\log n + \log \sigma)$ in IM.*

In each round we activate ranks r s.t. $\text{LF}(r)$ gets set in this round while r itself has not already been set in a previous round. We keep a bit vector S in external memory marking the indices of ranks for which we already observed the corresponding LCP value in a previous round. Remember that a rank l' gets set on S in the first round in which l' appears as a result interval lower bound of a call to $\text{BACKSTEP}(a, (l, r))$ for any arguments a, l and r . The result intervals for the BACKSTEP operation are encoded in the sequence \mathcal{Z} in the algorithm in Figure 3 after it has been sorted in line 21. Interval start points are marked by such tuples which have a non zero count (second component) attached. The information whether or not a rank will be newly set in S in the current round is encoded in the sequence \mathcal{Z}' in lines 22 – 25 of the algorithm. We perform an inverse LF mapping on \mathcal{Z}' by performing an inverse sorting of \mathcal{Z}' using BWT as key sequence. This allows us to determine which ranks need to be activated by combining information

from the sequence S and Z' during a linear scan of the two sequences (lines 28 – 32). The active set can be stored as a bit vector marking active ranks. The algorithm produces the indices of newly activated ranks in increasing order, so merging them into the already existing set is trivially performed in linear time $O(n)$. We keep the encoding of the PD vector from the previous section. Updating it by incrementing the counts for active ranks is straight forward and takes time $O(n)$. Finally the algorithm cleans the active set, sets the new ranks in S and computes the input intervals for the next round in lines 34 – 41. Again all of this is easily performed in time $O(n)$. The space usage in internal memory is reduced to $O(\log n + \log \sigma)$ (plus what is necessary to allow buffering for external memory).

Observe that in the reordering of values from rank to position order in the previous section the part taking the most IM is step 2. This is $O(\sigma \log n)$ bits. This is again caused by keeping track of the B function for each symbol of the alphabet while scanning the BWT to compute an LF mapping. As described above we can perform this LF mapping in EM while using $O(\log n + \log \sigma)$ in IM without asymptotically using more space in EM or time. This leads us to the following result.

Theorem 2. *The succinct $2n$ bit PLCP representation for a string s of length n can, given its BWT and sampled suffix array of sampling rate $\mathfrak{f} \in O(\log n)$, be constructed in worst cast time $O(n^2 \log \sigma)$ and average time $O(n \log n \log \sigma)$ using $O(n \log \sigma)$ bits of space in EM and $O(\log \sigma + \log n)$ bits of space in IM.*

6 Improvement of Worst Case

While on average our algorithm has a run time of $O(n \log n \log \sigma)$ as the LCP values are $O(\log n)$ on average, we often see cases in practice where, while most of the LCP values are small (in the order of $\log n$), there are some significantly larger values as well. In this case an easy adaption of our algorithm is to stop the computation of the PD vector after a certain number of rounds (say $3 \log n$) and compute the missing values using the algorithm presented in [15]. This adaption can be performed using the following steps before reordering the PD bit vector.

1. Erase all zero bits from the PD bit vector corresponding to ranks which are still in the active set. This removes incomplete values from PD for such ranks r where $\text{LCP}[r]$ was not yet reached but $\text{LCP}[\text{LF}(r)]$ was. This filtering takes time $O(n)$.
2. Compute a list S_{im} (irreducible missing) of ranks r in S s.t. $r = 0$ or $r > 0$ and $\text{BWT}[r - 1] \neq \text{BWT}[r]$ in time $O(n)$ and space $O(n)$ bits of EM. In the following let $n_{im} = |S_{im}|$.

```

PLCPEXTERNAL(BWT,  $n$ , ISA)
1  ( $Q, S$ )  $\leftarrow$  ( $\emptyset$ , bit vector of  $n$  false bits)
2   $Q$ .ENQUEUE( $(0, n)$ )
3  while  $|\{i \mid S_i = \text{true}\}| < n$  do
4       $Z \leftarrow$  empty sequence
5      while  $Q$ .HASNEXT() do
6           $(r_l, r_h) \leftarrow Q$ .NEXT()
7           $\triangleright$  extract sub sequence of BWT for interval  $[r_l, r_h)$ 
8           $A \leftarrow \text{BWT}_{r_l} \text{BWT}_{r_l+1} \dots \text{BWT}_{r_h-1}$ 
9          sort  $A$  in time  $O(|A| \log \sigma)$ 
10          $\triangleright$  attach count to first occurrence of each symbol and append to  $Z$ 
11          $\ell \leftarrow 0$ 
12         while  $\ell < r_h - r_l$  do
13              $(h, a) \leftarrow (\ell + 1, A[\ell])$ 
14              $\triangleright$  find end of range for same symbol
15             while  $h < r_h - r_l$  and  $A_h = a$  do
16                  $h \leftarrow h + 1$ 
17              $Z$ .APPEND( $(a, h - \ell)$ )
18             for  $i \leftarrow 1$  to  $(h - \ell) - 1$  do
19                  $Z$ .APPEND( $(a, 0)$ )
20              $\ell \leftarrow h$ 
21         sort  $Z$  by symbol component in time  $O(n \log \sigma)$ 
22          $\triangleright$  construct bit vector  $Z'$  marking ranks which will get set in this round
23         for  $r \leftarrow 0$  to  $n - 1$  do
24              $(a, c) \leftarrow Z_r$ 
25              $Z'_r \leftarrow (c \neq 0 \text{ and } S_r = \text{false})$ 
26          $\triangleright$  perform  $\text{LF}^{-1}$  mapping on  $Z'$ 
27         inverse sort  $Z'$  using BWT
28          $\triangleright$  activate ranks for this round
29         for  $r \leftarrow 0$  to  $n - 1$  do
30              $\triangleright$  if rank  $r$  not yet set but  $\text{LF}(r)$  will be set in this round
31             if  $Z'_r = \text{true}$  and  $S_r = \text{false}$  then
32                 activate  $r$ 
33         increment count for active ranks
34          $\triangleright$  update  $S$  and active set, construct intervals for next round
35          $NQ \leftarrow \emptyset$ 
36         for  $r'_l \leftarrow 0$  to  $n - 1$  do
37              $(a, c) \leftarrow Z_{r'_l}$ 
38             if  $c \neq 0$  then
39                 deactivate  $r'_l$  and set  $S_{r'_l}$ 
40                  $NQ$ .ENQUEUE( $(r'_l, r'_l + c)$ )
41      $Q \leftarrow NQ$ 

```

Fig. 3. Low internal memory variant PLCPexternal

3. Compute the list S_{imlf} containing the ranks in S_{im} and in addition for each rank $r \in S_{im}$ also $LF(r)$. This takes time $O(n \log \sigma)$ and space $O(n \log \sigma)$ in EM where we use a scan over BWT and a subsequent sorting by a symbol component as described above for computing the LF function for a set of ranks. This step adds all ranks for the previous position of a rank in S_{im} , which we need for computing differences between PLCP values for positions p in S_{im} and the respective previous positions $p - 1$.
4. For each rank $r > 0$ in S_{imlf} add $r - 1$ to S_{imlf} in time $O(n)$. We need these ranks for computing LCP values because $LCP[r]$ is defined by comparing the suffixes at the ranks r and $r - 1$.
5. Convert S_{imlf} to block code using $O(\log n)$ bits per rank in time $O(n)$ and space $O(n_{im} \log n)$ bits in EM.
6. Given a sampled inverse suffix array of sampling rate $f \in O(\log n)$ use a method similar to reordering the PLCP difference values above to annotate each rank in S_{imlf} with the corresponding position in time $O(n \log n \log \sigma)$ and space $O(n \log \sigma + n_{im} \log n)$ bits in EM.
7. Sort the resulting tuples by rank in time $O(n_{im} \log n)$ and space $O(n_{im} \log n)$ bits in EM.
8. For each (r, p) in the tuples s.t. there is some tuple $(r - 1, p')$ construct $(r, p = SA[r], r - 1, p' = SA[r - 1])$ in time $O(n_{im})$ and space $O(n_{im} \log n)$ EM bits.
9. Annotate the tuples with the respective LCP value between rank r and $r - 1$ stored in block code using a sparse version the algorithm presented in [15]. This requires the text s , which, if necessary, can be reconstructed from the BWT and an inverse sampled suffix array at sampling rate $f \in O(\log n)$ in time $O(n \log n \log \sigma)$ and space $O(n \log \sigma)$ in EM. Given $M \in O(n)$ words of IM (i.e. $O(M \log n)$ bits) of IM this requires time $O(\frac{n^2}{M \log \sigma n} + n \log \frac{M}{B} \frac{n}{B})$ using a disk block size of B words (see [15]). Drop the $r - 1$ and $p' = SA[r - 1]$ components from the tuples.
10. Sort the tuples by position. Drop all tuples for positions p s.t. $p > 0$ and there is no tuple for $p - 1$. For the rest replace the LCP component by the difference of the values for p and $p - 1$ if $p > 0$.
11. Sort the tuples by rank (time $O(n_{im} \log n)$) and insert the computed values into the PD bit vector (time $O(n)$).

Using this hybrid algorithm we can obtain a trade off between the faster worst case run time of the algorithm presented in [15] given sufficient IM and the reduced EM space usage of our algorithm presented above. In this second stage of the hybrid algorithm we are generally only interested in computing values for so called irreducible LCP values (cf. [16]) as only such values produce 0 bits in the succinct PLCP vector. The sum over all

irreducible LCP values for any string of length n is bounded by $2n \log n$ (see [16]). This bound is reached for de Bruijn strings (cf. [16]), however in this setting each irreducible LCP value is $\Theta(\log n)$. If we run the algorithm from the previous Section 4 for $O(\log^2 n)$ rounds, then all LCP values which remain unset must have a value of $\Omega(\log^2 n)$, which means there are $O(\frac{n}{\log n})$ such values and consequently the hybrid algorithm runs in worst case time $O(n \log^2 n \log \sigma)$ while using $O(n \log \sigma)$ space in EM and $O(\frac{n}{\log n})$ bits in IM.

Theorem 3. *Given the BWT and sampled inverse suffix array of sampling rate $\frac{1}{f} \in O(\log n)$ for a string s of length n over an alphabet of size σ the succinct permuted LCP array for s can be computed in time $O(n \log^2 n \log \sigma)$ while using $O(n \log \sigma)$ bits of space in EM and $O(\frac{n}{\log n})$ bits of space in IM.*

As the bound of $2n \log n$ for the sum over the irreducible LCP values of a string is obtained for LCP values which are all of length $O(\log n)$ the interesting question remains whether there is a smaller upper bound for the sum of the irreducible LCP values when only LCP values in $\omega(\log n)$ are considered in the sum.

7 Circular strings

In this section we relax the original requirement of a unique terminator symbol in s , i.e. we no longer require that $s_{n-1} < s_i$ for all $i < n - 1$. Let $\hat{s} = \hat{s}_0 \hat{s}_1 \dots$ be the infinite string defined by $\hat{s}_i = s_{i \bmod n}$. Further let $\hat{s}[i..] = \hat{s}_i \hat{s}_{i+1} \dots$ for $i \geq 0$, i.e. the suffix of \hat{s} starting from index i . We define that for two indices i, j the relation $\hat{s}[i..] < \hat{s}[j..]$ holds if either there is some l s.t. $\hat{s}_{i+l} < \hat{s}_{j+l}$ or $\hat{s}[i..] = \hat{s}[j..]$ and $i < j$. According to this definition we either have $\hat{s}[i..] < \hat{s}[j..]$ or $\hat{s}[j..] < \hat{s}[i..]$ for $i \neq j$ and in consequence there is a unique permutation $\hat{\mathbf{SA}} = \hat{\mathbf{SA}}_0, \hat{\mathbf{SA}}_1, \dots, \hat{\mathbf{SA}}_{n-1}$ of $0, 1, \dots, n - 1$ s.t. $\hat{s}[\hat{\mathbf{SA}}_{i-1}..] < \hat{s}[\hat{\mathbf{SA}}_i..]$ for $0 < i < n$ and we can define $\hat{\mathbf{BWT}}[i] = \hat{s}_{\hat{\mathbf{SA}}_i + n - 1}$. When defining a longest common prefix array for circular strings we face the issue of identical suffixes even when they start at different indices and thus infinite values in the array. These (infinite values) obviously occur in exactly such cases when s is an integer power of a string shorter than s (i.e. there is some string w s.t. $s = ww \dots w$ which we write as w^k if s consists of k copies of w juxtaposed). This case is easily detectable by scanning the BWT and determining whether there is some k dividing n s.t. for each i in $0, 1, \dots, \frac{n}{k} - 1$ we have $\hat{\mathbf{BWT}}[ik + 0] = \hat{\mathbf{BWT}}[ik + 1] = \dots = \hat{\mathbf{BWT}}[ik + k - 1]$. Figure 4 shows a linear time algorithm for detecting the maximum period p of s s.t. $s = s[0..p - 1]^{\frac{n}{p}}$. For obtaining a meaningful LCP array for a string $s = w^e$ for $e > 1$ we may choose to shrink it's $\hat{\mathbf{BWT}}$

```

DETECTPERIOD( $\hat{\mathbf{BWT}}, n$ )
1   $(e, i) \leftarrow (\infty, 0)$ 
2  while  $e > 1$  and  $i < n$  do
3       $(j, c) \leftarrow (i + 1, \hat{\mathbf{BWT}}[i])$ 
4      while  $j < n$  and  $c = \hat{\mathbf{BWT}}[j]$  do
5           $j \leftarrow j + 1$ 
6      if  $e = \infty$  then
7           $(e, i) \leftarrow (j - i, j)$ 
8      else  $(e, i) \leftarrow (\text{GCD}(j - i, e), j)$ 
9  return  $\frac{n}{e}$ 

```

Fig. 4. Linear time algorithm for detecting maximum period p s.t. the string of length n underlying $\hat{\mathbf{BWT}}$ equals $w^{\frac{n}{p}}$ for some word w

array to that of a single base factor w by keeping every e 'th symbol and discarding the symbols at the other indices.

In the following we assume that s is not an integer power of a word shorter than s and has length $n > 1$, i.e. s contains at least two different distinct symbols. As shown above this implies that for $0 \leq i < j < n$ there is always some $0 \leq l < n$ s.t. $\hat{s}_{i+l} \neq \hat{s}_{j+l}$. In consequence there is a well defined array $\hat{\mathbf{LCP}} = \hat{\mathbf{LCP}}_0, \hat{\mathbf{LCP}}_1, \dots, \hat{\mathbf{LCP}}_{n-1}$ given by $\hat{\mathbf{LCP}}_0 = 0$ and $\hat{\mathbf{LCP}}_i = l$ for $i = 1, 2, \dots, n-1$ where l is the smallest number s.t. $\hat{s}_{\hat{\mathbf{SA}}_{i-1}+l} \neq \hat{s}_{\hat{\mathbf{SA}}_i+l}$. Note that setting $\hat{\mathbf{LCP}}_0 = 0$ is consistent with the scheme for the other ranks as the suffixes at ranks 0 and $n-1$ start with different symbols, i.e. the length of their longest common prefix is 0. This also guarantees that the $\hat{\mathbf{LCP}}$ array contains the value 0 at least once. Based on the arrays $\hat{\mathbf{SA}}$ and $\hat{\mathbf{LCP}}$ we can define the array $\hat{\mathbf{ISA}}$ of length n by $\hat{\mathbf{ISA}}_{\hat{\mathbf{SA}}_i} = i$ for $i = 0, 1, \dots, n-1$ and $\hat{\mathbf{PLCP}} = \hat{\mathbf{PLCP}}_0, \hat{\mathbf{PLCP}}_1, \dots, \hat{\mathbf{PLCP}}_{n-1}$ by $\hat{\mathbf{PLCP}}_i = \hat{\mathbf{LCP}}_{\hat{\mathbf{ISA}}_i}$. The property of $\hat{\mathbf{PLCP}}_i - \hat{\mathbf{PLCP}}_{i-1} \geq -1$ still holds with the same arguments as in the non circular case, in fact this can even be extended to $\hat{\mathbf{PLCP}}_0 - \hat{\mathbf{PLCP}}_{n-1} \geq -1$ as the position 0 has no special meaning in the circular case. Note however that we loose one feature crucial for the $2n$ bit succinct PLCP representation in the transition to circular strings and this is the guarantee of $\hat{\mathbf{PLCP}}_{n-1} = 0$ which stems from the unique terminator symbol ensuring that no other suffix relevant for the computation of $\hat{\mathbf{LCP}}$ starts with the same symbol as the one at position $n-1$. As an example consider the string **abbab** with the $\hat{\mathbf{PLCP}}$ array 2, 1, 0, 0, 3 which would translate to the bit vector 0001110100001 of length 13 $> 10 = 2n$. Note that given $\hat{\mathbf{SA}}$ and a select dictionary on the bit vector we can correctly decode the respective $\hat{\mathbf{LCP}}$ values, however the vector is too long for the $2n$ bit bound. The reason for the excessive length is precisely the fact that the $\hat{\mathbf{PLCP}}$ array does not end with a 0 value. If we start off with the word **babba** which is a rotation of

abbab and consequently has the same $\hat{\text{BWT}}$ then the $\hat{\text{PLCP}}$ array is rotated to 3, 2, 1, 0, 0 with the bit vector 0000111101 of length $10 = 2n$. We chose **babba** because it shifts the positions by 1 from **abbab** and thus moves the last 0 at position $n - 2$ in the $\hat{\text{PLCP}}$ array of **abbab** to position $n - 1$ in the array for **babba**. We can obtain $\hat{\text{PLCP}}_i$ for **abbab** by decoding $\hat{\text{PLCP}}_{(i+1) \bmod n}$ for **babba** from the succinct PLCP bit vector for **babba**. Suitable ranks \hat{r} s.t. $\hat{\text{LCP}}_{\hat{r}} = 0$ can be found by checking the D array. Having chosen one such rank \hat{r} we can deduce the respective position \hat{p} by using a sampled inverse suffix array and the BWT in time $O(n \log \sigma \log n)$ if the sampling rate is $O(\log n)$ while using $O(n \log \sigma)$ bits in EM and $O(\log \sigma + \log n)$ bits in IM.

For computing the succinct PLCP bit vector of a string using $\hat{\text{BWT}}$ and a sampled inverse suffix array observe that the algorithm we presented in Section 4 and 5 has no knowledge about positions until it reaches the stage of reordering the values from rank to position order. All the generated values are purely differential (i.e. $\hat{\text{PLCP}}_i - \hat{\text{PLCP}}_{(i+n-1) \bmod n}$ for $0 \leq i < n$), in particular there is no special handling of position 0. The algorithm produces the bit vector 1110100001 for the input **abbab** which we need to rotate to 0000111101 as described above to obtain correct PLCP values while taking the employed position shift into account during decoding. The hybrid algorithm can also be adapted for circular strings without asymptotically modifying its runtime or space usage. In step 9. we need to take care of the fact that the comparison of two suffixes may extend beyond the end of s . Due to our pre conditions however we can guarantee that the longest common prefix of two different suffixes is always shorter than n symbols. This means that two runs over the set of blocks the text is decomposed into in the original algorithm are sufficient, where in the second run no more tuples are added but we only handle such tuples where the comparison extends across block boundaries. When accessing the text we need to use its circular extension for comparisons. In step 10. we need to handle the pair of positions $(n - 1, 0)$ if both positions are present. Asymptotically we keep the same time bound for the hybrid algorithm as we extend the amount of work done in step 9. by a constant factor 2 and in step 10. by a finite amount. This gives us the following result.

Theorem 4. *Given the circular $\hat{\text{BWT}}$ and sampled inverse suffix array of sampling rate $f \in O(\log n)$ for a circular string \hat{s} deduced from a string s of length n over an alphabet of size σ the succinct permuted LCP array for \hat{s} can be computed in time $O(n \log^2 n \log \sigma)$ while using $O(n \log \sigma)$ bits of space in EM and $O(\frac{n}{\log n})$ bits of space in IM.*

For the sake of this theorem the succinct permuted LCP array denotes the shifted version plus respective position shift described above. If the input

string s is an integer power of a shorter string s' s.t. s' is not itself an integer power of a shorter string, then the succinct permuted LCP array is constructed using s' .

References

1. G. Adelson-Velsky and E. Landis. An Algorithm for the Organization of Information. *Doklady Akademii Nauk USSR*, 146(2):263–266, 1962.
2. T. Beller, S. Gog, E. Ohlebusch, and T. Schnattinger. Computing the longest common prefix array based on the burrows-wheeler transform. *J. Discrete Algorithms*, 18:22–31, 2013.
3. T. Bingmann, J. Fischer, and V. Osipov. Inducing suffix and lcp arrays in external memory. In P. Sanders and N. Zeh, editors, *Proceedings of the 15th Meeting on Algorithm Engineering and Experiments, ALENEX 2013, New Orleans, Louisiana, USA, January 7, 2013*, pages 88–102. SIAM, 2013.
4. M. Burrows and D. Wheeler. A block-sorting lossless data compression algorithm. Technical Report 124, Digital Equipment Corporation, 1994.
5. M. Crochemore, C. Hancart, and T. Lecroq. *Algorithms on Strings*. Cambridge University Press, Cambridge, UK, 2007. 392 pages.
6. R. Dementiev, J. Kärkkäinen, J. Mehnert, and P. Sanders. Better external memory suffix array construction. *ACM Journal of Experimental Algorithmics*, 12:3.4:1–3.4:24, 2008.
7. P. Elias. Universal codeword sets and representations of the integers. *IEEE Transactions on Information Theory*, 21(2):194–203, 1975.
8. P. Ferragina, T. Gagie, and G. Manzini. Lightweight data indexing and compression in external memory. *Algorithmica*, 63(3):707–730, 2012.
9. P. Ferragina and G. Manzini. Opportunistic data structures with applications. In *41st Annual Symposium on Foundations of Computer Science, FOCS 2000, 12-14 November 2000, Redondo Beach, California, USA*, pages 390–398. IEEE Computer Society, 2000.
10. P. Ferragina and G. Manzini. An experimental study of a compressed index. *Inf. Sci.*, 135(1-2):13–28, 2001.
11. P. Ferragina, G. Manzini, V. Mäkinen, and G. Navarro. An alphabet-friendly fm-index. In A. Apostolico and M. Melucci, editors, *String Processing and Information Retrieval, 11th International Conference, SPIRE 2004, Padova, Italy, October 5-8, 2004, Proceedings*, volume 3246 of *Lecture Notes in Computer Science*, pages 150–160. Springer, 2004.
12. R. Grossi, A. Gupta, and J. S. Vitter. High-order entropy-compressed text indexes. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, January 12-14, 2003, Baltimore, Maryland, USA.*, pages 841–850. ACM/SIAM, 2003.
13. R. Grossi and J. S. Vitter. Compressed suffix arrays and suffix trees with applications to text indexing and string matching (extended abstract). In F. F. Yao and E. M. Luks, editors, *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*, pages 397–406. ACM, 2000.
14. W. Hon, K. Sadakane, and W. Sung. Breaking a time-and-space barrier in constructing full-text indices. *SIAM J. Comput.*, 38(6):2162–2178, 2009.
15. J. Kärkkäinen and D. Kempa. LCP array construction in external memory. In J. Gudmundsson and J. Katajainen, editors, *Experimental Algorithms - 13th International Symposium, SEA 2014, Copenhagen, Denmark, June 29 - July 1, 2014*.

- Proceedings*, volume 8504 of *Lecture Notes in Computer Science*, pages 412–423. Springer, 2014.
16. J. Kärkkäinen, G. Manzini, and S. J. Puglisi. Permuted longest-common-prefix array. In G. Kucherov and E. Ukkonen, editors, *Combinatorial Pattern Matching, 20th Annual Symposium, CPM 2009, Lille, France, June 22-24, 2009, Proceedings*, volume 5577 of *Lecture Notes in Computer Science*, pages 181–192. Springer, 2009.
 17. J. Kärkkäinen, P. Sanders, and S. Burkhardt. Linear work suffix array construction. *J. ACM*, 53(6):918–936, 2006.
 18. T. Kasai, G. Lee, H. Arimura, S. Arikawa, and K. Park. Linear-time longest-common-prefix computation in suffix arrays and its applications. In A. Amir and G. M. Landau, editors, *Combinatorial Pattern Matching, 12th Annual Symposium, CPM 2001 Jerusalem, Israel, July 1-4, 2001 Proceedings*, volume 2089 of *Lecture Notes in Computer Science*, pages 181–192. Springer, 2001.
 19. U. Manber and E. W. Myers. Suffix arrays: A new method for on-line string searches. *SIAM J. Comput.*, 22(5):935–948, 1993.
 20. J. I. Munro. Tables. In V. Chandru and V. Vinay, editors, *Foundations of Software Technology and Theoretical Computer Science, 16th Conference, Hyderabad, India, December 18-20, 1996, Proceedings*, volume 1180 of *Lecture Notes in Computer Science*, pages 37–42. Springer, 1996.
 21. D. Okanohara and K. Sadakane. A linear-time burrows-wheeler transform using induced sorting. In J. Karlgren, J. Tarhio, and H. Hyrö, editors, *String Processing and Information Retrieval, 16th International Symposium, SPIRE 2009, Saariselkä, Finland, August 25-27, 2009, Proceedings*, volume 5721 of *Lecture Notes in Computer Science*, pages 90–101. Springer, 2009.
 22. K. Sadakane. New text indexing functionalities of the compressed suffix arrays. *J. Algorithms*, 48(2):294–313, 2003.
 23. K. Sadakane. Compressed suffix trees with full functionality. *Theory Comput. Syst.*, 41(4):589–607, 2007.
 24. W. Szpankowski. On the Height of Digital Trees and Related Problems. *Algorithmica*, 6(1-6):256–277, 1991.
 25. G. Tischler. Faster average case low memory semi-external construction of the Burrows-Wheeler Transform. In C. S. Iliopoulos and A. Langiu, editors, *Proceedings of the 2nd International Conference on Algorithms for Big Data, Palermo, Italy, April 07-09, 2014.*, volume 1146 of *CEUR Workshop Proceedings*, pages 61–68. CEUR-WS.org, 2014.
 26. G. Tischler. Faster Average Case Low Memory Semi-External Construction of the Burrows-Wheeler Transform. *Mathematics in Computer Science*, accepted 2014.
 27. P. Weiner. Linear pattern matching algorithms. In *14th Annual Symposium on Switching and Automata Theory, Iowa City, Iowa, USA, October 15-17, 1973*, pages 1–11. IEEE Computer Society, 1973.